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A note on the uniform asymptotic expansion of integrals with coalescing endpoint and saddle points

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Abstract. For the uniform asymptotic expansion of certain types of contour integrals, one of whose critical points is an endpoint of the interval of integration, a method alternative to Bleistein's is introduced and numerically tested by way of a non-trivial physical example.

The analysis of a variety of physical problems eventually requires the suitable approximation of contour integrals of the form

$$I(\omega; \boldsymbol{\alpha}; \zeta_0) = \int_C f(\zeta; \boldsymbol{\alpha}) \, \mathrm{e}^{\mathrm{i}\omega F(\zeta; \boldsymbol{\alpha})} \, \mathrm{d}\zeta, \tag{1}$$

for $\omega \to \infty$, where for all admissible values of the set of real parameters $\boldsymbol{\alpha}$, $f(\zeta; \boldsymbol{\alpha})$ and $F(\zeta; \boldsymbol{\alpha})$ are analytic functions along the contour of integration C, which has at least one finite endpoint ζ_0. The other endpoint is either also finite, or it is, more frequently, at infinity. Numerous applications require that $f(\zeta; \boldsymbol{\alpha})$ and $F(\zeta; \boldsymbol{\alpha})$ be allowed to be multivalued and to be given on multisheet Riemann surfaces, on which an appropriate arrangement of branch cuts is assumed to have been fixed. In these cases, the second endpoint may also be a branch point of the integrand. When the second endpoint is not a regular finite point, we assume that $\mathrm{Re}[\mathrm{i}F(\zeta; \boldsymbol{\alpha})] \to -\infty$ as that endpoint is approached along C.

It is well known that the behaviour of (1) for large values of ω depends crucially on the number, type and distribution of the so-called 'critical points' of its integrand (Bleistein and Handelsman 1975). One such configuration that has emerged particularly frequently in the analysis of physical problems comprises, apart from the endpoint of integration ζ_0, two simple saddle points of F at ζ_n, $n = 1, 2$, so that

$$[\mathrm{d}F(\zeta; \boldsymbol{\alpha})/\mathrm{d}\zeta]|_{\zeta_n} = 0. \tag{2}$$

Since in all physical applications F is real for real ζ and $\boldsymbol{\alpha}$, we may infer from Schwarz's reflection principle,

$$F(\zeta^*; \boldsymbol{\alpha}) = F^*(\zeta; \boldsymbol{\alpha}), \tag{3}$$

that the saddle points appear as a complex conjugate pair,

$$\zeta_1 = \zeta_2^* = \zeta_s. \tag{4}$$

As $\boldsymbol{\alpha}$ varies, these two saddle points are allowed to coalesce and to approach indefinitely the endpoint of integration ζ_0.

The necessity of finding uniform asymptotic approximations to the integral (1) under the specifications just given has arisen in a number of optical diffraction problems in the short wavelength limit (Levey and Felsen 1969, Orlov 1975, 1976), in acoustical diffraction problems in the short wavelength limit (Marston and Langley 1983), in the study of the long time limit of the propagation and compression of FM pulses in dispersive media (Felsen 1971) and in the investigation of the long time effect of small random perturbations on deterministic systems with two stable nodes and one saddle point as their stationary states (Mangel 1979). Recently, a generalised version of this problem, where more than one pair of saddle points may coalesce with the endpoint of integration ζ_0, has appeared in the accurate determination of the spectral and angular distribution of synchrotron radiation emitted from planar strong field wigglers with arbitrary field variation (Leubner and Ritsch 1985).

The now standard method of deriving uniform asymptotic approximations to integrals of the type (1) was introduced by Bleistein (1967), and subsequently refined by Ursell (1972). It consists of two steps (Bleistein and Handelsman 1975). The first of these is the local transformation of (1) to the so-called canonical form by means of the mapping

$$\mathrm{i}F(\zeta;\boldsymbol{\alpha})=P(z;z_{\mathrm{s}}^{(i)}),\tag{5}$$

where in the case at hand P is suitably chosen as a polynomial of degree three,

$$P(z;z_{\mathrm{s}})=\tfrac{1}{3}z^3-z_{\mathrm{s}}^2z+\gamma,\tag{6}$$

with its two real saddle points at $\pm z_{\mathrm{s}}$ being the images of the original saddle points at $\zeta_{\mathrm{s}}, \zeta_{\mathrm{s}}^*$ under the mapping (6). From (3) and (4), we immediately find

$$z_{\mathrm{s}}=\{\tfrac{3}{2}\,\mathrm{Im}[F(\zeta_{\mathrm{s}};\boldsymbol{\alpha})]\}^{1/3}$$

and

$$\gamma=\mathrm{i}\,\mathrm{Re}[F(\zeta_{\mathrm{s}};\boldsymbol{\alpha})].$$

(More detailed discussions of this mapping can be found, for example, in Chester *et al* (1957) and Bleistein and Handelsman (1975), and for two coalescing pairs of saddle points in Leubner (1981) and Connor *et al* (1984).) The integral (1) then becomes

$$I(\omega;\boldsymbol{\alpha};\zeta_0)=\int_{C_z}f(\zeta;\boldsymbol{\alpha})\frac{\mathrm{d}\zeta}{\mathrm{d}z}\,\mathrm{e}^{\omega P}\,\mathrm{d}z,\tag{7}$$

where C_z is the (local) image of the original contour C, extending from z_0—being the image of ζ_0 under (6)—to infinity, the latter fact being a consequence of our assumption on the behaviour of $\mathrm{Re}[F(\zeta;\boldsymbol{\alpha})]$ along the contour C.

In a second step, Bleistein (1967) expands

$$f(\zeta;\boldsymbol{\alpha})\,\mathrm{d}\zeta/\mathrm{d}z=a_0+b_0z+(\mathrm{d}P/\mathrm{d}z)g_0(z).\tag{8}$$

Upon insertion into (7), this allows us to integrate by parts,

$$I(\omega;\boldsymbol{\alpha};\zeta_0)=a_0\Gamma_0+b_0\Gamma_1-\frac{1}{\omega}g_0(z_0)\,\mathrm{e}^{\omega P(z_0)}-\frac{1}{\omega}\int_{C_z}\frac{\mathrm{d}g_0}{\mathrm{d}z}\,\mathrm{e}^{\omega P(z)}\,\mathrm{d}z,\tag{9}$$

where

$$\Gamma_n=\int_{C_z}z^n\,\mathrm{e}^{\omega P}\,\mathrm{d}z,\qquad n=0,1,\tag{10}$$

are called the canonical integrals for the problem in hand, which are here incomplete Airy functions (Levey and Felsen 1969). Next, one expands

$$\mathrm{d}g_0(z)/\mathrm{d}z = a_1 + b_1 z + g_1(z)\,\mathrm{d}P/\mathrm{d}z,$$

and, continuing as above, one arrives at an asymptotic expansion of (1), as Bleistein (1967) and Ursell (1972) have shown.

This is the method invoked in the work of Levey and Felsen (1969), which in turn has been cited in all the physical applications listed above.

However, for a number of reasons it is worthwhile to point out that there exists an alternative method for the derivation of the asymptotic expansion in question, which does not seem to have been given before.

What we are looking for is an appropriate series expansion of the left-hand side of (8), analogous to the series expansion employed by Chester *et al* (1957) in their pioneering work, where for $z_0 \to \infty$ they used

$$f(\zeta;\boldsymbol{\alpha})\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \sum_{n=0}^{\infty}(a_n + b_n z)\left(\frac{\mathrm{d}P}{\mathrm{d}z}\right)^n. \tag{11}$$

The point is that while Bleistein's (1967) method is perfectly suited for demonstrating that with its help one indeed obtains an asymptotic expansion of (7), the advantage of a series expansion of the form (11) is that the process of determining the expansion coefficients and that of repeatedly integrating by parts are completely separated. If higher terms of the pertinent asymptotic expansion are required, this facilitates the implementation of this alternative approach on a computer, which is a point of great practical importance since the determination of higher terms of an asymptotic series tends to become rather involved (Dingle 1973, Leubner 1981).

Although Levey and Felsen (1969) claim to have made use of the series (11) in deriving the leading terms of the asymptotic expansion (9), they have in fact not done so, and could not have, because in the case at hand this series would be inadequate. This can be immediately inferred from the fact that the function $g_0(z)$, appearing in (8) and in the lowest-order part of (9), involves coefficients of (11) to arbitrarily high order. In contrast, an adequate series expansion of the left-hand side of (11) must be such that a certain order of the corresponding asymptotic series involves only a finite number of its coefficients.

For the currently considered case of a single pair of saddle points that can coalesce with ζ_0, such a series expansion is provided by

$$f(\zeta;\boldsymbol{\alpha})\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \sum_{n=0}^{\infty}[A_n + B_n(z-z_0) + C_n(z-z_0)^2]\left((z-z_0)\frac{\mathrm{d}P}{\mathrm{d}z}\right)^n. \tag{12}$$

If, on the other hand, we had to take account of four saddle points that could coalesce with ζ_0, P on the right-hand side of (5) would be of degree five (Leubner 1981, Connor *et al* 1984), and (12) would have to be replaced by

$$\begin{aligned} f(\zeta;\boldsymbol{\alpha})\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \sum_{n=0}^{\infty}&[A_n + B_n(z-z_0) + C_n(z-z_0)^2 \\ &+ D_n(z-z_0)^3 + E_n(z-z_0)^4][(z-z_0)\,\mathrm{d}P/\mathrm{d}z]^n, \end{aligned} \tag{13}$$

with suitable generalisations of (13) applying for other saddle point configurations of interest.

Returning to (12), we first observe that, in view of

$$\mathrm{d}P/\mathrm{d}z = (z-z_0)^2 + 2z_0(z-z_0) + (z_0^2 - z_s^2) \tag{14}$$

being a polynomial of degree two in $(z-z_0)$, both the expansions (11) and (12) are just particular rearrangements of the Taylor series of $f(\zeta;\boldsymbol{\alpha})(\mathrm{d}\zeta/\mathrm{d}z)$ around z_0, which by assumption exists in some neighbourhood of this point. But, as an essential difference, we note that the factor $[(z-z_0)\,\mathrm{d}P/\mathrm{d}z]^n$ in the nth term of (12) permits, after insertion of (12) into (7), a certain number $m \leqslant n$ of successive integrations by parts, where, in contrast to an analogous procedure based on (11), the integrated term vanishes at *both* ends of the contour of integration. Since each integration by parts multiplies the nth term by a factor of ω^{-1}, this term contributes at most to order $\mathrm{O}(\omega^{-m})$ to the resulting asymptotic series. For example, it is easy to show that the third term of (12) permits two overall integrations by parts and thus contributes at most to order $\mathrm{O}(\omega^{-2})$. We further note that, whenever $(z-z_0)^2$ occurs, it can be expressed through (14), so that the asymptotic series stemming from (12) also involves only the *two* canonical integrals (10) that we encountered in the above sketch of Bleistein's (1967) method.

Explicitly, we find up to order $\mathrm{O}(\omega^{-2})$ by the method described

$$
\begin{aligned}
I(\omega;\boldsymbol{\alpha};\zeta_0) \simeq \Gamma_0\{&A_0 - z_0B_0 + (z_0^2+z_s^2)C_0 + \omega^{-1}[-A_1 + 2z_0B_1 - 3(z_0^2+z_s^2)C_1 \\
&+ 2z_0(z_0^2+z_s^2-1)A_2 - 2(z_0^4+3z_0^2z_s^2-z_0^2-z_s^2)B_2 \\
&+ 2z_0(z_0^4+6z_0^2z_s^2+z_s^4-z_0^2-3z_s^2)C_2]\} \\
&+ \Gamma_1\{B_0 - 2z_0C_0 + \omega^{-1}[-2B_1 + 6z_0C_1 - 2(z_0^2+z_s^2)A_2 \\
&+ 2z_0(z_0^2+3z_s^2)B_2 - 2(z_0^4+6z_0^2z_s^2+z_s^4)C_2]\} \\
&+ [\mathrm{e}^{\omega P(z_0)}/\omega]\{-C_0 + \omega^{-1}[3C_1 - 2z_0A_2 + 2(z_0^2+z_s^2)B_2 \\
&- 2z_0(z_0^2+3z_s^2)C_2]\},
\end{aligned}
\tag{15}
$$

which, as it should be, can be shown to be identical with the expansion following from Bleistein's (1967) method.

In a particular application of (15), any triple of coefficients A_n, B_n, C_n in (12) is found by differentiating (12) n times with respect to z, and then setting $z=z_0$, $z=z_s$ and $z=-z_s$, respectively, with a straightforward modification applying in the degenerate case $z_0=z_s=0$.

The numerical accuracy of the expansion (15) and of its generalisations to more complicated saddle point configurations was thoroughly tested by way of the class of integrals

$$
I(\omega;\boldsymbol{\alpha};0) = \int_C \mathrm{d}\zeta\,\mu(\zeta)\exp\left[\mathrm{i}\frac{\omega}{c}\int^{\zeta}\left(\frac{1+n_1\mu(\bar{\zeta})}{[\beta_0^2-\mu^2(\bar{\zeta})]^{1/2}} - n_3\right)\mathrm{d}\bar{\zeta}\right],
\tag{16}
$$

which arose in the quoted investigation (Leubner and Ritsch 1985).

These integrals involve the (non-vanishing component of the) vector potential $\mu(\zeta)$ of the particular (planar) wiggler-magnet arrangement under consideration, the injection velocity $\beta_0 \leqslant 1$ of the relativistic electron beam along the wiggler axis, and the direction $(n_1, n_2, n_3) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ under which the synchrotron signal is observed. In the argument of the exponential, that branch of the square root must be taken that is real and positive for a real and positive radicand. The contour C depends on both $\mu(\zeta)$ and the set of parameters chosen, and is specified as that path extending from the origin of the ζ plane (that is, $\zeta_0=0$) along which $\mathrm{Im}[\mathrm{i}F(\zeta;\boldsymbol{\alpha})]$ remains constant, and $\mathrm{Re}[\mathrm{i}F(\zeta;\boldsymbol{\alpha})]$ is (monotonically) decreasing.

For two reasons the integrals (16) are ideally suited for demonstrating the usefulness of asymptotic expansions of the type (15) in general, and the need for expansions beyond the leading terms in particular. Firstly, the argument $i\omega F(\zeta;\, \boldsymbol{\alpha})$ of the exponential in (16) is itself an integral that cannot be expressed in terms of standard functions in all cases of physical interest except the one considered in (18) below. As a consequence, the asymptotic expansion (15) and its generalisations to more complicated saddle point configurations are not only considerably easier to evaluate than the original expression (16), but they also furnish a much better insight into the dependence of (16) on its parameters. Secondly, the value of $I(\omega;\, \boldsymbol{\alpha};\, 0)$ is required for such a range of the parameter ω that an approximation by only the leading terms of its asymptotic expansion would be insufficient. In physical terms, this corresponds to the fact that one wants to predict the synchrotron spectrum for all frequencies that are radiated with appreciable intensity, and these range from relatively small to very large values of ω.

From (5), the saddle points of F in (16) are determined by

$$\mu(\zeta) = -\frac{n_1}{n_1^2 + n_3^2} \pm i\frac{n_3(\gamma_0^{-2} + \beta_0^2 n_2^2)^{1/2}}{n_1^2 + n_3^2}, \tag{17}$$

where $\gamma_0 = (1 - \beta_0^2)^{-1/2}$ has been used. As is to be expected from (3) and (4), they appear in complex conjugate pairs, but the number of pairs and their mutual arrangement depends on the choice of the vector potential $\mu(\zeta)$.

In the simplest possible case, this function is chosen as

$$\mu(\zeta) = \kappa\zeta / L, \tag{18}$$

with L being the length of one wiggler segment, and the constant $|\kappa|$ being small compared with β_0 for a realistic experiment. There is only one pair of saddle points, the positions ζ_s and ζ_s^* of which are given by $(L/|\kappa|)$ times the right-hand side of (17). Inspection reveals that for the physically interesting case of $n_3 > 0$, where we observe the synchrotron signal at acute angles with respect to the relativistic electron beam, they are situated on the sheet that also contains the contour of integration C. For $n_3 \to 0+$, they approach opposite lips of the branch cut stemming from the square root in the exponent of (16), and they move across the cut into the lower sheet for n_3 turning negative. Since $n_1 = n_2 = 0$ and $L/(|\kappa|\gamma_0) \ll 1$ are admissible (and physically interesting) parameter ranges, it is obvious that the pair of saddle points may indefinitely approach the endpoint $\zeta_0 = 0$ of the contour of integration. For a typical set of parameters, figure 1(a) shows the (μ- and parameter-dependent) contour C, the location of the two saddle points and the two branch points with an appropriate choice of the corresponding branch cuts.

Table 1 compares the errors committed by representing (16), with the choice (18) of the vector potential, by only the leading terms of its asymptotic expansion (that is, up to $O(\omega^{-1})$), and by the approximation (15) (that is, up to $O(\omega^{-2})$), respectively, relative to a high precision numerical integration of (16) as it stands. In view of the many parameters that could be varied in this integral, it was of course necessary to restrict the table to the results of a small sample, but checks for a great variety of other such combinations showed the corresponding errors to follow pretty much the same pattern.

Table 1 reflects two important points. The first of these is to be expected, namely, that for all values of ω shown, the expansion (15) brings about a considerably better approximation to the integral (16) than that furnished by merely the leading terms of

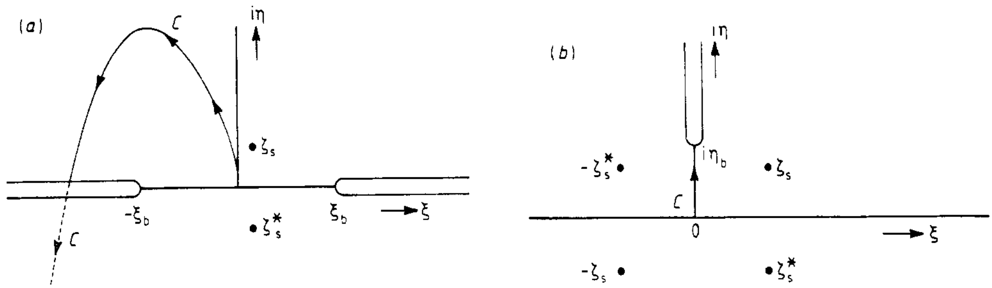

Figure 1. (*a*) Saddle points ζ_s and ζ_s^*, branch points ξ_b and $-\xi_b$, and corresponding branch cuts of $F(\zeta;\ \boldsymbol{\alpha})$, and contour of integration C, for the integral (16) with the function $\mu(\zeta)$ and the following parameter values. $\mu(\zeta)=\kappa\zeta/L$, $\kappa=-0.3$, $L=2$, $\beta_0=0.990$, $n_1=n_2=0.2$. (*b*) Saddle points ζ_s, $-\zeta_s$, ζ_s^* and $-\zeta_s^*$, branch point $\mathrm{i}\eta_b$, and corresponding branch cut of $F(\zeta;\ \boldsymbol{\alpha})$, and contour of integration C, for the integral (16) with the function $\mu(\zeta)$ and the following parameter values. $\mu(\zeta)=\kappa[1-\cos(2\pi\zeta/L)]$, $\kappa=-0.3$, $L=2$, $\beta_0=0.990$, $n_1=n_2=0.2$. $\zeta=\xi+\mathrm{i}\eta$.

Table 1. Comparison of the relative errors in % committed by representing the integral equation (16) with (18) asymptotically to first and second order. Throughout, κ, β_0 and n_2 have been chosen to be -0.6, 0.99 and 10^{-3}, respectively, and $\omega_0=[F(L/2;\ \boldsymbol{\alpha})]^{-1}$, with $L=1$.

		Relative error of asymptotic expansion up to order	
$10^3\times n_1$	ω/ω_0	$O(\omega^{-1})$	$O(\omega^{-2})$
2.5	8	1.6391	0.5597
	16	1.2134	0.1966
	32	1.0457	0.0422
4.5	8	1.6444	0.5510
	16	1.2247	0.1941
	32	1.0651	0.0470
6.5	8	1.6494	0.5422
	16	1.2340	0.1980
	32	1.0787	0.0480
8.5	8	1.6545	0.5334
	16	1.2432	0.1847
	32	1.0907	0.0479

the pertinent asymptotic series. The second point is that down to rather small values of ω, the expansion (15) alone suffices to represent the original integral uniformly to within a satisfactory numerical accuracy.

Another physically interesting case is characterised by the vector potential

$$\mu(\zeta)=\kappa[1-\cos(2\pi\zeta/L)], \qquad (19)$$

which in view of $\beta_0/|\kappa|>1$ gives rise to branch points of $F(\zeta;\ \boldsymbol{\alpha})$ at ζ_b, determined by

$$\cos(2\pi\zeta_b/L)=1\pm\beta_0/\kappa,$$

one of which is situated at $\zeta_b=\mathrm{i}\eta_b$, say, $\eta_b>0$, on the imaginary axis of the ζ plane.

The corresponding contour turns out to extend from $\zeta = 0$ to the branch point $\zeta_b = i\eta_b$, which at first sight may seem to render the formalism inapplicable that leads to an expansion of the type (15). However, since $\mathrm{Re}[iF(\zeta; \boldsymbol{\alpha})] \to -\infty$ as $\zeta \to i\eta_b$, we arrive at practically the same numerical value for $I(\omega; \boldsymbol{\alpha}; 0)$ by terminating the integration at $i(\eta_b - \delta)$, $\delta \ll 1$, so that now $F(\zeta; \boldsymbol{\alpha})$ is analytic everywhere on this 'truncated' contour of integration, in keeping with the requirements formulated above.

Insertion of (19) into (17) shows that there are now two pairs of saddle points, which for appropriate parameter values may indefinitely approach the origin of the ζ plane and thus the endpoint of the contour of integration C. These four saddle points require the replacement of (12) by (13) and, by arguments completely analogous to those leading from (12) to (15), we arrive at an asymptotic expansion of the type (15) that, in contrast to (15), involves four canonical integrals Γ_n, $n = 0, 1, 2, 3$. These are given by (10), if P is interpreted as a polynomial of degree five, and could be named, with reference to Connor *et al* (1984), *incomplete* canonical swallowtail integrals.

As should be expected, a comparison analogous to that made in table 1 shows this series to be as satisfactory an asymptotic representation for the case of four saddle points coalescing with an endpoint of the contour of integration, as is (15) for the case of two such saddle points, and similar results can be anticipated for even more general cases. For reasons given, the series expansions (12) and (13) and their generalisations, therefore, constitute a useful alternative to Bleistein's (1967) method.

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